Lecture 7: Countable and uncountable sets

Definition 1: A set S is called denumerable if it is equivalent to $\mathbb{N}$.

2: A set is called countable if it is either finite or denumerable.

3: A set which is not countable is called uncountable set.

Examples:

1. The set of all even natural number E is countable by $f : \mathbb{N} \to E$ as $f(n) = 2n$.
2. The set $S = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$ is countable by $f : \mathbb{N} \to S$ as $f(n) = \frac{n}{(n+1)}$ for all $n \in \mathbb{N}$.
3. $\mathbb{Z}$ is countable by $f : \mathbb{Z} \to \mathbb{N}$ as

$$f(n) = \begin{cases} 2n & \text{if } n > 0 \\ 1 - 2n & \text{if } n \leq 0. \end{cases}$$

Lemma: If a set X is infinite, then there exists a one-one function $f : \mathbb{N} \to X$.
Proof: Let X be infinite. Then $\exists$ an element say $a_1 \in X$. We show by induction that for every $n \geq 2$, $\exists\, a_n \in X$ different from $a_1, \ldots, a_{n-1}$.
Now, a_1 has been chosen, consider the set $X \setminus \{a_1\}$. If this set is empty, then $X = \{a_1\}$, which is finite. As X is infinite $X \setminus \{a_1\}$ is nonempty, so let $a_2 \in X \setminus \{a_1\}$. This proves the basis case.
So suppose $a_1, \ldots, a_m \in X$ has been chosen corresponding to the numbers $1, 2, \ldots, m$, the set $X \setminus \{a_1, \ldots, a_m\}$ is non-empty, otherwise $X = \{a_1, \ldots, a_n\}$ would be finite. So let $a_{m+1} \in X \setminus \{a_1, a_2, \ldots, a_m\}$. This proves the induction steps.
Hence, corresponding to 1, there exists $a_1 \in X$, and for each $n \geq 2$, there exists $a_n \in X$ different from all of $a_1, \ldots, a_{n-1}$. Define the function $f : \mathbb{N} \to X$ by $f(n) = a_n$. Then f is one-one.

Theorem: For a non-empty set A following statements are equivalent:

1. A is countable
2. There is a surjective map from $\mathbb{N}$ to A
3. There is an injective map from A to $\mathbb{N}$

proof (1) $\implies$ (2). Suppose that A is countable. There are two cases-

- A is countably infinite
- A is finite

When A is countably infinite then $A \approx \mathbb{N}$. There exists a bijective map $f\colon A \to \mathbb{N}$, which is also sujective. When A is finite then since $A \neq \phi$, so $A \approx J_n = \{1, 2, \ldots, n\}$ for some positive integer n. That means there is a bijective map $g\colon J_n \to A$. We define a map $h\colon \mathbb{N} \to A$ by

$$h(k) := \begin{cases} g(k) & \text{if } k = 1, 2, \ldots, n \\ g(1) & \text{otherwise} \end{cases}.$$

Thus $h(\mathbb{N}) = g(J_n) = A$. So the map h is surjective.
(2) $\implies$ (3). Assume (2) occurs. That means there exist a surjective map $f\colon \mathbb{N} \to A$. We wish to find an injective map from A to $\mathbb{N}$. Since f is surjective, for any $a \in A, f^{-1}(a) = \{x \in \mathbb{N} | f(x) = a\}$ is a non-empty subset of $\mathbb{N}$. By well ordering property of $\mathbb{N}$, $f^{-1}(a)$ has least element for every $a \in A$, which is unique. We define a map $g\colon A \to \mathbb{N}$ by $g(a) =$ the least element of $f^{-1}(a)$ for every $a \in A$. Clearly when $a \neq b$ then $g(a) \neq g(b)$ because $f^{-1}(a) \cap f^{-1}(b) = \phi$. This proves (3).
(3) $\implies$ (1). Suppose $f : A \to \mathbb{N}$ is injective map. If A is finite, then nothing to prove. Suppose A is an infinite. Then by above lemma, there exists injective map $g : \mathbb{N} \to A$. Now by CSB-theorem, there exists a bijective map $h : A \to \mathbb{N}$. So A is countable.

1. Let X and Y be sets and let $f : X \to Y$ be an injective map. If Y is countable then so is X.
   Proof: Since Y is countable, There is a bijective map $g : Y \to \mathbb{N}$. Then the function $g \circ f : X \to \mathbb{N}$ is injective. Hence by above theorem, X is countable.

2. A subset of a countable set is countable.
   Proof: Let A be a countable set and $S \subseteq A$. Since A is countable, $\exists$ an injective map $f : A \to \mathbb{N}$. Also inclusion map $i : S \to A$ is injective. Then the composition map $f \circ i : S \to \mathbb{N}$ is an injective map. Hence S is countable.

3. The image of a countable set under any map is countable.
   Proof: Let $f\colon A \to B$ be a surjective map, where A is a countable set. Since A is countable so there exists a surjective map from $g\colon \mathbb{N} \to A$. Considering the composite map $fog\colon \mathbb{N} \to B$ is surjective as composition of two surjective maps is surjective. Hence B is countable.

4. The product of two countable sets is countable.
   Proof: Let A and B be countable sets. Then there exist bijective maps $f\colon \mathbb{N} \to A$ and $g\colon \mathbb{N} \to B$. Define a map $h\colon \mathbb{N} \times \mathbb{N} \to A \times B$ by $h(m, n) = (f(m), g(n))$. Clearly h is a bijection. Also since $\mathbb{N} \approx \mathbb{N} \times \mathbb{N}$, so $\mathbb{N} \approx A \times B$.

5. $\mathbb{Q}$ is countable.
   Proof: Define $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ by $f(\frac{a}{b}) = (a, b)$, here $g.c.d.(a, b) = 1$. Then f is injective. Also, we get a bijective map $g : \mathbb{Z} \times \mathbb{N} \to \mathbb{N}$. Then the map $g \circ f : \mathbb{Q} \to \mathbb{N}$ is injective and hence by the above lemma, $\mathbb{Q}$ is countable.

Theorem: The union of two countable sets is countable.
Proof: Let A and B be countable set. We may assume without loss of generality that A and B are disjoint. We can do this since $A \cup B = A \cup (B \setminus A)$, and Since $B \setminus A \subseteq B$ therefore it is also countable. Let $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$ be bijective maps. Define $h : \mathbb{N} \to A \cup B$ by $h(n) = \begin{cases} f(k) & \text{if}\, n = 2k \\ g(k) & \text{if}\, n = 2k - 1 \end{cases}$.

Then h is surjective. Hence by above theorem $A \cup B$ is countable.

Theorem A countable union of countable sets is countable is countable.
Proof: Let $\{A_i\}_{i\in\mathbb{N}}$ be a countable family, where each A_i is countable. Let $X = \cup_{i\in\mathbb{N}} A_i$.
If X is finite then nothig to prove. So assume that X is not finite. Then by above lemma there exists an injective map $f : \mathbb{N} \to X$.
Let $x \in X$. Then there exists at least one $i \in \mathbb{N}$ such that $x \in A_i$. Since A_i is countable, let x appears at the k-th place in the enumeration of A_i.
Thus corresponding to each $x \in X$, we have a unique pair (i, k) of natural numbers. Now define $g : X \to \mathbb{N}$ by $g(x) = 2^i 3^k$, where i is the smallest natural number such that $x \in A_i$ and x appears at k-th position in the enumeration of A_i. Note that g is one-one. Hence by CSB theorem X is countable.

Theorem For any $k \in \mathbb{N}$, the Cartesian product $\mathbb{N}^k$ is denumerable.
Proof: Note that the function $f : \mathbb{N} \to \mathbb{N}^k$ given by $f(m) = (m, 1, \dots, 1)$ is one-one.
Let $p_1, p_2, \dots, p_k$ be the first k number of primes. Define $g : \mathbb{N}^k \to \mathbb{N}$ by $g(m_1, m_2, \dots, m_k) = p_1^{m_1-1}.p_2^{m_2-1}.\dots.p_k^{m_k-1}$. Then g is one-one. Now by CSB theorem $\mathbb{N}^k$ is denumerable.

Theorem A finite product of countable set is countable.
Proof: Let $A_1, \dots, A_k$ be countable sets. We want to show that $X = A_1 \times \dots \times A_k$ is countable. If any $A_i =$, then $X =$. So assume that each A_i is nonempty. Since A_i is nonempty, there exists a one-one function $f_i : A_i \to \mathbb{N}$. Then the function $f : X \to \mathbb{N}^k$ defined by $f(x_1, \dots, x_k) = (f_1(x_1), \dots, f_k(x_k))$ is one-one. Let $g : \mathbb{N}^k \to \mathbb{N}$ be one-one function defined by $g(m_1, m_2, \dots, m_k) = p_1^{m_1-1}.p_2^{m_2-1}.\dots.p_k^{m_k-1}$. Then $g \circ f : X \to \mathbb{N}$ is one-one. Hence X is countable.

Remark: The above result is not true for infinite product. For example if $S := \{0, 1\}^{\mathbb{N}}$, then S is not countable. Although infinite product of a singleton set is countable.

Proof: Consider the set of all sequence on $\{0, 1\}$ i.e., $\prod_{n=1}^{\infty}\{0, 1\}$

$$S = \{f | f : \mathbf{N} \to \{0, 1\}\}$$

we will prove set S is uncountable. if it is countable then $\exists$ a enumaration of elements of S, as follows

$$f_i = (a_{i1}, a_{i2}, ..., a_{in}, ...) \qquad \forall i \in \mathbf{N}$$

let us construct a sequence f' in {0,1} as follows.

$$f' = f'_i \qquad \forall i \in \mathbf{N}$$

such that $f'_i = 0$ if $f_{ii} = 1$ & $f'_i = 1$ if $f_{ii} = 0$. clearly this,

$$f' \notin S$$

this shows that it is not possible to enumerate the elements of S.

Theorem: $\mathbb{R}$ is uncountable.
Proof: Suppose $\mathbb{R}$ is countable. We know that a subset of a countable set is countable.

consider $A = (0,\ 1) \subseteq \mathbb{R}$. We show that A is not countable. On the contrary, suppose A is countable. Then we can write elements of A as $r_1, r_2, r_3 \ldots$, where r_i can be written as $r_i = d_{i1}d_{i2}d_{i3}\ldots$, where $d_{ij} \in \{0, 1, 2, \ldots, 9\}$. Now consider $r = d_1d_2d_3\ldots$ as follows:

$$d_i = \begin{cases} 1 & d_{ii} \neq 1 \\ 2 & d_{ii} = 1. \end{cases}$$

Then $r \in A$ but not equal to r_i. Thus A is uncountable and hence $\mathbb{R}$.

Cantor's Theorem: There exists no surjection from a set X to its power set $\mathcal{P}(X)$.

Proof: On the contrary suppose $f : X \to \mathcal{P}(X)$ is an onto map. For eaxh $x \in X$, $f(x) \subseteq X$. Consider the set $Y = \{x \in X : x \notin f(x)\}$.
Since $Y \in \mathcal{P}(X)$ and f is onto, there exists $s \in X$ with $f(s) = Y$. Then we have two possibilities: $s \in Y$ and $s \notin Y$.
If $s \in Y$, then $s \notin f(s) = Y$. A contradiction.
If $s \notin Y$, then $s \in f(s) = Y$. A contradiction.